Introduction to Brownian Motion

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Abstract

This paper presents the basic knowledge of a standard Brownian motion which is a building block of all stochastic processes. A standard Brownian motion is a subclass of 1) continuous martingales, 2) Markov processes, 3) Gaussian processes, and 4) Itô diffusion processes. It is also a subclass of Lévy processes although we will not discuss this in this sequel.

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[1] Brownian Motion

[1.1] Standard Brownian Motion

Definition 1.1 Standard Brownian motion (Standard Wiener process) A standard Brownian motion $(B_{t\in[0,\infty)})$ is a real valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ satisfying:

(1) Its increments are independent. In other words, for $0 \le t_1 < t_2 < ... < t_n < \infty$:

$$\mathbb{P}(B_{t_0} \cap B_{t_1} - B_{t_0} \cap B_{t_2} - B_{t_1} \cap ... \cap B_{t_n} - B_{t_{n-1}})$$
$$= \mathbb{P}(B_{t_0})\mathbb{P}(B_{t_1} - B_{t_0})\mathbb{P}(B_{t_2} - B_{t_1})...\mathbb{P}(B_{t_n} - B_{t_{n-1}}).$$

(2) Its increments are stationary (time homogeneous): i.e. for $h \ge 0$, $B_{t+h} - B_t$ has the same distribution as B_h. In other words, the distribution of increments does not depend on t.

(3) $\mathbb{P}(B_0 = 0) = 1$. The process starts from 0 almost surely (with probability 1).

(4) $B_t \sim Normal(0, t)$. Its increments follow a Gaussian distribution with the mean 0 and the variance t.

Definition 1.2 Standard Brownian motion with starting point c Let c be a real valued constant or a random variable independent of a standard Brownian motion $(B_{t\in[0,\infty)})$. Then, a standard Brownian motion with starting point c is a real valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$:

$$(c + B_{t\in[0,\infty)}).$$

Theorem 1.1 Standard Brownian motion A standard Brownian motion process $(B_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ satisfies the following conditions:

(1) The process is stochastically continuous: $\forall \varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \ge \varepsilon) = 0$.

(2) Its sample path (trajectory) is continuous in t (i.e. continuous $\in$ rcll) almost surely.

Proof

Consult Karlin (1975). We have to remind you that this proof is not that simple.

[1.2] Brownian Motion with Drift

Definition 1.3 Brownian motion with drift Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, a Brownian motion with drift is a real valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ as:

$$(X_{t\in[0,\infty)}) \equiv (\mu t + \sigma B_{t\in[0,\infty)}),$$

where $\mu \in \mathbb{R}$ is called a drift and $\sigma \in \mathbb{R}^{+}$ is called a diffusion (volatility) parameter. A Brownian motion with drift satisfies the following conditions:

(1) Its increments are independent. In other words, for $0 \le t_1 < t_2 < ... < t_n < \infty$:

$$\mathbb{P}(X_{t_0} \cap X_{t_1} - X_{t_0} \cap X_{t_2} - X_{t_1} \cap ... \cap X_{t_n} - X_{t_{n-1}})$$
$$= \mathbb{P}(X_{t_0})\mathbb{P}(X_{t_1} - X_{t_0})\mathbb{P}(X_{t_2} - X_{t_1})...\mathbb{P}(X_{t_n} - X_{t_{n-1}}).$$

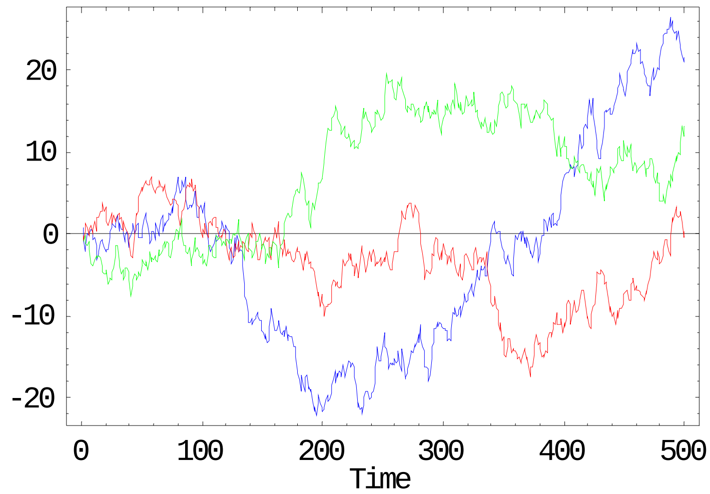
(2) Its increments are stationary (time homogeneous): i.e. for $h \ge 0$, $X_{t+h} - X_t$ has the same distribution as X_h. In other words, the distribution of increments does not depend on t.

(3) $X_t \equiv \mu t + \sigma B_t \sim Normal(\mu t, \sigma^2 t)$. Its increments follow a Gaussian distribution with the mean μt and the variance $\sigma^2 t$.

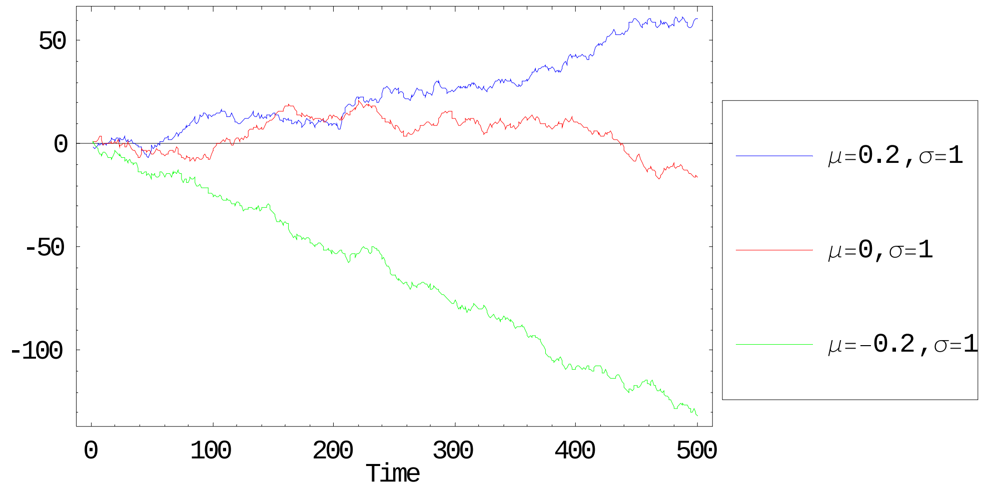
(4) Its sample path (trajectory) is continuous in t (i.e. continuous $\in$ rcll) almost surely.

[1.3] Sample Paths Properties of Brownian Motion

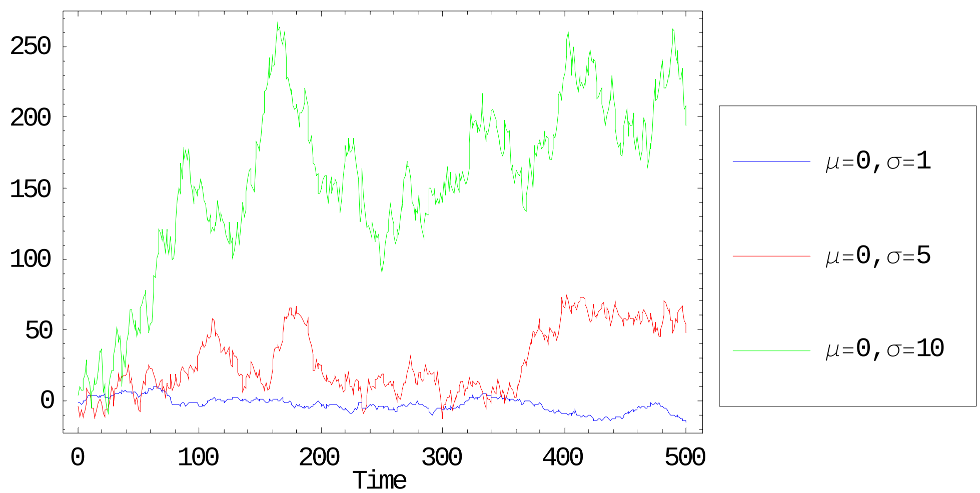
Before discussing the sample paths properties of Brownian motion, take a look at simulated sample paths of a standard Brownian motion on Panel (A) in Figure 1.1 and those of a Brownian motion with drift on Panel (B) and (C).



A) Sample Paths of a Standard Brownian Motion.



B) Sample Paths of a Brownian Motion with Drift. Different drifts and same diffusion parameters.



C) Sample Paths of a Brownian Motion with Drift. Zero drifts and different diffusion parameters.

Figure 1.1 Simulated Sample Paths of Brownian Motion

Theorem 1.2 Sample paths properties of Brownian motion with drift Consider a real valued Brownian motion with drift $(X_{t\in[0,T]}) \equiv (\mu t + \sigma B_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, the sample paths of $(X_{t\in[0,T]})$ possess following properties:

(1) Sample paths are continuous with probability 1.

(2) Sample paths are of infinite variation on any finite interval $[0,t]$. In other words, the total variation on any finite interval $[0,t]$ of a sample path of a Brownian motion with drift is infinite with probability 1 in the limit $n \to \infty$ (as the partition becomes finer and finer):

$$\mathbb{P}\left(\lim_{n\to\infty} T(X) = \lim_{n\to\infty} \sup \sum_{i=1}^{n} \left| X(t_i) - X(t_{i-1}) \right| = \infty \right) = 1 .$$

Intuitively speaking, the infinite variation property means highly oscillatory sample paths.
(3) The quadratic variations of sample paths of Brownian motions with drift $(X_{t\in[0,T]})$ are finite on any finite interval $[0,t]$ and converge to $\sigma^2 t$ with probability 1 in the limit $n \to \infty$ (as the partition becomes finer and finer):

$$\mathbb{P}\left(\lim_{n\to\infty} T^2(X) = \lim_{n\to\infty} \sup \sum_{i=1}^{n} \left| X(t_i) - X(t_{i-1}) \right|^2 = \sigma^2 t < \infty \right) = 1 .$$

For more details and proofs about theorem 1.2, consult Sato (1999) page 22 – 28 and Karatzas and Shreve (1991) section 1.5 and 2.9. We also recommend Rogers and Williams (2000) chapter 1.

[1.4] Equivalent Transformations of Brownian Motion

Theorem 1.3 Equivalent transformations of Brownian motion If $(B_{t\in[0,\infty)})$ is a real valued standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$, then, it satisfies the four conditions:

(1) A standard Brownian motion $(B_{t\in[0,\infty)})$ is symmetric. In other words, the process $(-B_{t\in[0,\infty)})$ is also a standard Brownian motion:

$$(B_{t\in[0,\infty)}) \stackrel{d}{=} (-B_{t\in[0,\infty)}) .$$

(2) A standard Brownian motion $(B_{t\in[0,\infty)})$ has a time shifting property. In other words, the process $(B_{t+A} - B_A)$ is also a standard Brownian motion for $\forall A \in \mathbb{R}^+$:

$$(B_{t+A} - B_A) \stackrel{d}{=} (B_{t\in[0,\infty)}) .$$

(3) Time scaling property of a standard Brownian motion. For any nonzero $c \in \mathbb{R}$, the process $(\sqrt{c} B_{t/c})$ or $(\frac{1}{\sqrt{c}} B_{ct})$ is also a standard Brownian motion:

$$(\frac{1}{\sqrt{c}} B_{ct}) \underset{=}{d} (\sqrt{c} B_{t/c}) \underset{=}{d} (B_{t\in[0,\infty)}).$$

(4) Time inversion property of a standard Brownian motion (i.e. a variant of (3)). The process defined as:

$$(\tilde{B}_{t\in[0,\infty)}) = \begin{cases} 0 & \text{if} \quad t = 0 \\ (tB_{1/t}) & \text{if } 0 < t < \infty \end{cases},$$

is also a standard Brownian motion:

$$(\tilde{B}_{t\in[0,\infty)}) \underset{=}{d} (B_{t\in[0,\infty)}).$$

Proof

These are easy exercises for readers. For the proof of the continuity of $(\tilde{B}_{t\in[0,\infty)})$ at 0, consult Rogers and Williams (2000) page 4.

[1.5] Characteristic Function of Brownian Motion

Consider a real valued Brownian motion with drift process $(X_{t\in[0,\infty)}) \equiv (\mu t + \sigma B_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Its characteristic function can be obtained by the direct use of the definition of a characteristic function (i.e. Fourier transform of the probability density function with Fourier transform parameters $(1,1)$):

$$\phi_{X_t}(\omega) \equiv \mathcal{F}\left[\mathbb{P}(x)\right] \equiv \int_{-\infty}^{\infty} e^{i\omega x} \mathbb{P}(x) dx$$

$$\phi_{X_t}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{ -\frac{\left(x - \mu t\right)^2}{2\sigma^2 t} \right\} dx$$

$$\phi_{X_t}(\omega) = \exp(i\mu t\omega - \frac{\sigma^2 t\omega^2}{2}).$$

[2] Brownian Motion as a Subclass of Continuous Martingale

Definition 2.1 Continuous martingale A continuous stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is said to be a continuous martingale with respect to the filtration $\mathcal{F}_t$ and under the probability measure $\mathbb{P}$ if it satisfies the following conditions:

(1) X_t is nonanticipating.

(2) $E[|X_t|] < \infty$ for $\forall t \in [0, T]$. Finite mean condition.
(3) $E[X_u | \mathcal{F}_t] = X_t$ for $\forall u > t$.

In other words, if a stochastic process is a martingale, then, the best prediction of its future value is its present value. Note that the definition of martingale makes sense only when the underlying probability measure P and the filtration $\mathcal{F}_t$ have been specified.

The fundamental property of a martingale process is that its future variations are completely unpredictable with the filtration $\mathcal{F}_t$:

$$\forall u > 0,\ E[x_{t+u} - x_t | \mathcal{F}_t] = E[x_{t+u} | \mathcal{F}_t] - E[x_t | \mathcal{F}_t] = x_t - x_t = 0 .$$

Finite mean condition (2) is necessary to ensure the existence of the conditional expectation.

[2.1] A Continuous Martingale Property of Standard Brownian Motion

Theorem 2.1 Standard Brownian motion is a continuous martingale Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, $(B_{t\in[0,\infty)})$ is a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

By definition, $(B_{t\in[0,\infty)})$ is a nonanticipating process (i.e. $\mathcal{F}_{t\in[0,\infty)}$ - adapted process) with the finite mean $E[|B_t|] = 0 < \infty$ for $\forall t \in [0,\infty)$. For $\forall 0 \le t \le u < \infty$:

$$B_u = B_t + \int_t^u dB_v . \tag{1}$$

Using the equation (1) and the fact that a Brownian motion is a nonanticipating process, i.e. $E[B_t | \mathcal{F}_t] = B_t$:

$$E[B_u - B_t | \mathcal{F}_t] = E[B_u | \mathcal{F}_t] - E[B_t | \mathcal{F}_t] = E[B_t + \int_t^u dB_v | \mathcal{F}_t] - B_t$$
$$E[B_u - B_t | \mathcal{F}_t] = E[B_t | \mathcal{F}_t] + E[\int_t^u dB_v | \mathcal{F}_t] - B_t$$
$$E[B_u - B_t | \mathcal{F}_t] = B_t + 0 - B_t = 0 ,$$

or in other words:

$$E[B_u | \mathcal{F}_t] = E[B_t + \int_t^u dB_v | \mathcal{F}_t] = E[B_t | \mathcal{F}_t] + E[\int_t^u dB_v | \mathcal{F}_t] = B_t + 0$$
$$E[B_u | \mathcal{F}_t] = B_t ,$$

which is a martingale condition.

□

Theorem 2.2 Squared standard Brownian motion B_t^2 is not a continuous martingale
Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, B_t^2 is not a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

Using the equation (1) and independent increments condition, for $0 \le t \le u < \infty$:

$$E[B_u^2 | \mathcal{F}_t] = E[(B_t + \int_t^u dB_v)^2 | \mathcal{F}_t]$$
$$E[B_u^2 | \mathcal{F}_t] = E[B_t^2 + 2B_t \int_t^u dB_v + \int_t^u dB_v^2 | \mathcal{F}_t]$$
$$E[B_u^2 | \mathcal{F}_t] = E[B_t^2 | \mathcal{F}_t] + E[2B_t \int_t^u dB_v | \mathcal{F}_t] + E[\int_t^u dB_v^2 | \mathcal{F}_t]$$
$$E[B_u^2 | \mathcal{F}_t] = E[B_t^2 | \mathcal{F}_t] + E[\int_t^u dB_v^2 | \mathcal{F}_t]$$
$$E[B_u^2 | \mathcal{F}_t] = B_t^2 + (u - t) ,$$

which violates a martingale condition.

□

Theorem 2.3 Squared standard Brownian motion minus time $B_t^2 - t$ is a continuous martingale Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, $B_t^2 - t$ is a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

Using the equation (1) and independent increments condition, for $0 \le t \le u < \infty$:

$$E[B_u^2 - u | \mathcal{F}_t] = E[(B_t^2 - t) + \{(\int_t^u dB_v)^2 - (u - t)\} | \mathcal{F}_t]$$

$$E[B_u^{\,2} - u \,|\, \mathcal{F}_t] = E[(B_t^{\,2} - t) \,|\, \mathcal{F}_t] + E[(\int_t^u dB_v)^2 - (u-t) \,|\, \mathcal{F}_t]$$
$$E[B_u^{\,2} - u \,|\, \mathcal{F}_t] = B_t^{\,2} - t + E[(\int_t^u dB_v)^2 \,|\, \mathcal{F}_t] - (u-t)$$
$$E[B_u^{\,2} - u \,|\, \mathcal{F}_t] = B_t^{\,2} - t + (u-t) - (u-t) = B_t^{\,2} - t \,,$$

which satisfies a martingale condition.

□

Theorem 2.4 Converse of theorem 2.3 Let $(X_{t\in[0,\infty)})$ be a continuous martingale defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, the process $(X_{t\in[0,\infty)})$ is a standard Brownian motion if and only if it satisfies:

(1) $X_0 = 0$. The process starts from zero.
(2) $X_t^{\,2} - t$ is a martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

[2.2] Nonmartingale Property of a Brownian Motion with Drift

Theorem 2.5 Brownian motion with drift is not a continuous martingale Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, a Brownian motion with drift $(X_{t\in[0,\infty)}) \equiv (\mu t + \sigma B_{t\in[0,\infty)})$ is not a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

By definition, $(X_{t\in[0,\infty)})$ is a nonanticipating process (i.e. $\mathcal{F}_{t\in[0,\infty)}$- adapted process) with the finite mean $E[X_t] = E[\mu t + \sigma B_t] = \mu t < \infty$ for $\forall t \in [0,\infty)$ and $\mu \in \mathbb{R}$. For $\forall 0 \le t \le u < \infty$:

$$X_u = X_t + \int_t^u dX_v \,. \tag{2}$$

Using the equation (2) and the fact that a Brownian motion with drift is a nonanticipating process, i.e. $E[X_t \,|\, \mathcal{F}_t] = X_t$:

$$E[X_u \,|\, \mathcal{F}_t] = E[X_t + \int_t^u dX_v \,|\, \mathcal{F}_t] = E[X_t \,|\, \mathcal{F}_t] + E[\int_t^u dX_v \,|\, \mathcal{F}_t]$$
$$E[X_u \,|\, \mathcal{F}_t] = X_t + \mu(u-t) \,,$$

which violates a martingale condition.

□

Theorem 2.6 Detrended Brownian motion with drift is a continuous martingale
Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, a detrended Brownian motion with drift defined as:

$$(X_{t\in[0,\infty)} - \mu t) \equiv (\mu t + \sigma B_{t\in[0,\infty)} - \mu t) \equiv (\sigma B_{t\in[0,\infty)}),$$

is a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

For $\forall 0 \le t \le u < \infty$:

$$E[X_u - \mu u | \mathcal{F}_t] = E[(X_t - \mu t) + (\int_t^u dX_v - \mu \int_t^u dv) | \mathcal{F}_t]$$
$$E[X_u - \mu u | \mathcal{F}_t] = E[(X_t - \mu t) | \mathcal{F}_t] + E[(\int_t^u dX_v - \mu \int_t^u dv) | \mathcal{F}_t]$$
$$E[X_u - \mu u | \mathcal{F}_t] = X_t - \mu t + \mu(u-t) - \mu(u-t)$$
$$E[X_u - \mu u | \mathcal{F}_t] = X_t - \mu t,$$

which satisfies a martingale condition.

□

[2.3] A Continuous Martingale Property of Exponential Standard Brownian Motion

Theorem 2.7 Exponential of a standard Brownian motion is a continuous martingale
Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, for any $\theta \in \mathbb{R}$, the exponential of a standard Brownian motion defined as:

$$Z_t = \exp(\theta B_t - \frac{1}{2}\theta^2 t), \qquad (3)$$

is a continuous martingale with respect to the filtration $\mathcal{F}_{t\in[0,\infty)}$ and the probability measure $\mathbb{P}$.

Proof

We first prove the often used proposition.

Proposition 2.1 Note that if $X \sim Normal(\mu t, \sigma^2 t)$, then for any $\theta \in \mathbb{R}$:

$$E[\exp(\theta X)] = \exp(\theta\mu t + \frac{1}{2}\theta^2\sigma^2 t) \,. \tag{4}$$

Proof

$$\begin{aligned}
E[\exp(\theta X)] &= \int_{-\infty}^{\infty} \exp(\theta X)\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{(X-\mu t)^2}{2\sigma^2 t}\}dX \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{-\theta X\, 2\sigma^2 t + X^2 - 2X\mu t + \mu^2 t^2}{2\sigma^2 t}\}dX \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{X^2 - 2(\theta\sigma^2 t + \mu t)X + \mu^2 t^2}{2\sigma^2 t}\}dX \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{(X-(\theta\sigma^2 t + \mu t))^2 - (\theta\sigma^2 t + \mu t)^2 + \mu^2 t^2}{2\sigma^2 t}\}dX \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{(X-(\theta\sigma^2 t + \mu t))^2}{2\sigma^2 t}\}\exp\{\frac{(\theta\sigma^2 t + \mu t)^2 - \mu^2 t^2}{2\sigma^2 t}\}dX \\
&= \exp\{\frac{(\theta\sigma^2 t + \mu t)^2 - \mu^2 t^2}{2\sigma^2 t}\}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{(X-(\theta\sigma^2 t + \mu t))^2}{2\sigma^2 t}\}dX \\
&= \exp\{\frac{(\theta\sigma^2 t + \mu t)^2 - \mu^2 t^2}{2\sigma^2 t}\} = \exp\{\frac{\theta^2\sigma^4 t^2 + 2\theta\sigma^2 t\mu t}{2\sigma^2 t}\} \\
&= \exp\{\theta\mu t + \frac{1}{2}\theta^2\sigma^2 t\}
\end{aligned}$$

□

Now we are ready to prove the Brownian exponential defined by the equation (3) is a martingale.

Firstly, the process $(Z_{t\in[0,\infty)})$ is nonanticipating because a standard Brownian motion $(B_{t\in[0,\infty)})$ is nonanticipating.

Secondly, it satisfies the finite mean condition, since $E[Z_t] = 1 < \infty$:

$$E[Z_t] = E[\exp(\theta B_t - \frac{1}{2}\theta^2 t)]$$

$$E[Z_t] = E[\exp(\theta B_t)\exp(-\frac{1}{2}\theta^2 t)]$$

$$E[Z_t] = \exp(-\frac{1}{2}\theta^2 t)E[\exp(\theta B_t)] \,,$$

using the proposition 2.1, $E[\exp(\theta B_t)] = \exp(\frac{1}{2}\theta^2 t)$:

$$E[Z_t] = \exp(-\frac{1}{2}\theta^2 t)\exp(\frac{1}{2}\theta^2 t) = 1.$$

For $\forall 0 \le t \le t+h < \infty$, by the definition of Z_t:

$$E[Z_{t+h} | \mathcal{F}_t] = E[\exp\{\theta B_{t+h} - \frac{1}{2}\theta^2 (t+h)\} | \mathcal{F}_t].$$

The trick is to multiply $\exp(\theta B_t - \theta B_t) = e^0 = 1$ inside the expectation operator:

$$E[Z_{t+h} | \mathcal{F}_t] = E[\exp(\theta B_t - \theta B_t)\exp\{\theta B_{t+h} - \frac{1}{2}\theta^2 (t+h)\} | \mathcal{F}_t]$$
$$E[Z_{t+h} | \mathcal{F}_t] = E[\exp(\theta B_t)\exp(-\theta B_t)\exp(\theta B_{t+h})\exp(-\frac{1}{2}\theta^2 t)\exp(-\frac{1}{2}\theta^2 h) | \mathcal{F}_t]$$
$$E[Z_{t+h} | \mathcal{F}_t] = E[\exp(\theta B_t - \frac{1}{2}\theta^2 t)\exp\{\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\} | \mathcal{F}_t].$$

Since Brownian increments are independent:

$$E[Z_{t+h} | \mathcal{F}_t] = E[\exp(\theta B_t - \frac{1}{2}\theta^2 t) | \mathcal{F}_t] E[\exp\{\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\} | \mathcal{F}_t],$$

and since B_t is $\mathcal{F}_t$-adapted:

$$E[Z_{t+h} | \mathcal{F}_t] = \exp(\theta B_t - \frac{1}{2}\theta^2 t) E[\exp\{\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\}].$$

By the definition of Z_t:

$$E[Z_{t+h} | \mathcal{F}_t] = Z_t E[\exp\{\theta(B_{t+h} - B_t)\}\exp(-\frac{1}{2}\theta^2 h)],$$

and since $\exp(-\frac{1}{2}\theta^2 h)$ is a constant:

$$E[Z_{t+h} | \mathcal{F}_t] = Z_t \exp(-\frac{1}{2}\theta^2 h) E[\exp\{\theta(B_{t+h} - B_t)\}].$$

Use the proposition 2.1 because $B_{t+h} - B_t \sim Normal(0, h)$:

$$E[Z_{t+h} | \mathcal{F}_t] = Z_t \exp(-\frac{1}{2}\theta^2 h)\exp(\frac{1}{2}\theta^2 h)$$
$$E[Z_{t+h} | \mathcal{F}_t] = Z_t .$$

□

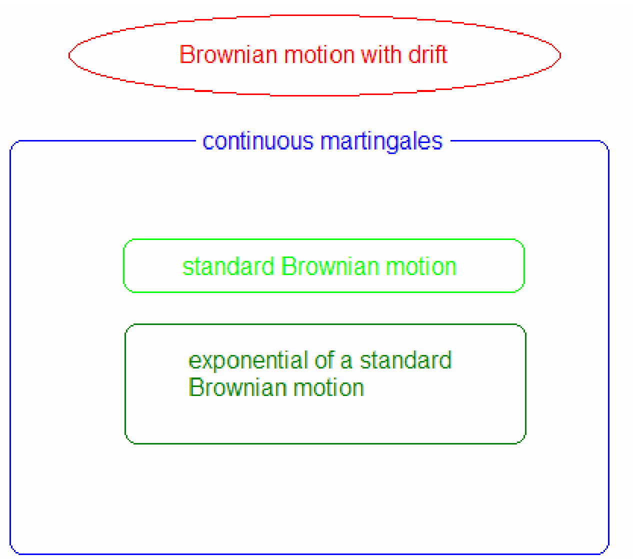


Figure 2.1 Brownian motion as a subclass of continuous martingales

[3] Brownian Motion as a Subclass of Gaussian Processes

Definition 3.1 Gaussian process A stochastic process $(X_{t\in[0,\infty)})$ on $\mathbb{R}^d$ (i.e. this means that X_t is a d -dimensional vector) defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is said to be a Gaussian process, if, for any increasing sequence of time $0 \le t_1 < t_2 < ... < t_k < \infty$, the law of any finite dimensional vector $\left(X(t_1), X(t_2), ..., X(t_k) \right)$ of the process is multivariate normal.

Because all finite dimensional multivariate normal distributions are uniquely determined by their means and covariance function, Gaussian processes can be defined in an alternate way.

Definition 3.2 Gaussian process A stochastic process $(X_{t\in[0,\infty)})$ on $\mathbb{R}^d$ (i.e. this means that X_t is a d -dimensional vector) defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is said to be a Gaussian process, if the law of the process $(X_{t\in[0,\infty)})$ is uniquely determined by:

(1) Means $E[X_t]$.

(2) Covariance functions $Cov(X_t, X_u) = E[\{X_t - E(X_t)\}\{X_u - E(X_u)\}^T]$ for $\forall 0 \leq t \neq u < \infty$,

where T is a transposition operator.

Theorem 3.1 A standard Brownian motion A standard Brownian motion $(B_{t\in[0,\infty)})$ is a one dimensional Gaussian process with:

(1) Zero mean $E[B_t] = 0$.
(2) Covariance function $Cov(B_t, B_u) = t \wedge u = \min\{t,u\}$.

Proof

By the definition of a standard Brownian motion. For more details, we recommend Karlin and Taylor (1975) page 376-377.

The converse of the theorem 3.1 is also true.

Theorem 3.2 A standard Brownian motion A real valued one dimensional Gaussian stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is a standard Brownian motion with drift, if its mean and covariance function satisfy:

(1) $E[X_t] = 0$.
(2) $Cov(X_t, X_u) = t \wedge u = \min\{t,u\}$

Proof

By the definition of a standard Brownian motion.

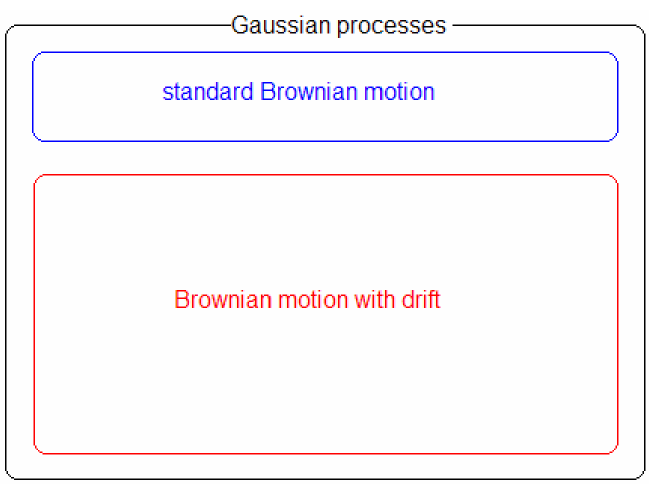


Figure 3.1 Brownian motion as a subclass of Gaussian processes

[4] Brownian Motion as a Subclass of Markov Processes

We first introduce the definitions and terminologies used in the study of Markov processes.

Definition 4.1 Transition function Consider a continuous time nonanticipating stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ which takes values in a measurable space $(B, \mathcal{B})$ (i.e. $B \in \mathcal{B}(\mathbb{R})$). $(B, \mathcal{B})$ is called a state space of the process and the process is said to be B - valued. Consider an increasing sequence of time $0 \le t \le u \le v < \infty$. A real valued transition function $\mathbb{P}_{t,v}(x, B)$ with $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ is a mapping which satisfies the following conditions:

(1) $\mathbb{P}_{t,v}(x, B)$ is a probability measure which maps every fixed x into B.
(2) $\mathbb{P}_{t,v}(x, B)$ is $\mathcal{B}$ - measurable for every $B \in \mathcal{B}(\mathbb{R})$.
(3) $\mathbb{P}_{t,t}(x, B) = \delta(B)$.
(4) $\mathbb{P}_{t,v}(x, B) = \int_{\mathbb{R}} \mathbb{P}_{t,u}(x, dy)\mathbb{P}_{u,v}(y, B)$.

The condition (4) is called the Chapman-Kolmogorov identity. Chapman-Kolmogorov identity means that the transition probability $\mathbb{P}_{t,v}(x, B)$ of moving from a state x at time t to a state B at time v can be calculated as a sum (i.e. integral) of the product of the transition probabilities via an intermediate time $t \le u \le v$, i.e. $\mathbb{P}_{t,u}(x, dy)$ and $\mathbb{P}_{u,v}(y, B)$. In the general cases, transition functions are dependent on the states and time.

Definition 4.2 Time homogeneous (temporary homogeneous or stationary) transition function Consider an increasing sequence of time $0 \le t \le u \le v < \infty$. A real valued transition function $\mathbb{P}_{t,v}(x, B)$ with $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ is said to be time homogeneous if it satisfies:

$$\mathbb{P}_{t,v}(x, B) = \mathbb{P}_{0,v-t}(x, B) = \mathbb{P}_{v-t}(x, B),$$

which indicates that the transition function $\mathbb{P}_{t,v}(x, B)$ of moving from a state x at time t to a state B at time v is equivalent to the transition function $\mathbb{P}_{0,v-t}(x, B)$ of moving from a state x at time 0 to a state B at time $v - t$. In other words, the transition function is independent of the time t and depends only on the interval of time $v - t$.

Definition 4.3 Chapman-Kolmogorov identity for the time homogeneous transition function Consider an increasing sequence of time $0 \le t \le u < \infty$. Chapman-Kolmogorov identity for the time homogeneous transition function is:

$$\int_{\mathbb{R}} \mathbb{P}_{0,t}(x, dy)\mathbb{P}_{0,u}(y, B) = \int_{\mathbb{R}} \mathbb{P}_{t}(x, dy)\mathbb{P}_{u}(y, B) = \mathbb{P}_{t+u}(x, B).$$

Definition 4.4 Markov Processes (less formal) Consider a continuous time nonanticipating stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Then, the process $(X_{t\in[0,\infty)})$ is said to be a Markov process if it satisfies, for every increasing sequence of time $0 < t_1 \leq t_2 \leq ... \leq t_n \leq t \leq u < \infty$:

$$\mathbb{P}(X_u \left| \mathcal{F}_t \right.) = \mathbb{P}(X_u \left| X_0, X_{t_1}, X_{t_2}, ..., X_{t_n}, X_t \right.) = \mathbb{P}(X_u \left| X_t \right.),$$

Informally, Markov property means that the probability of a random variable X_u at time $u \geq t$ (tomorrow) conditional on the entire history of the stochastic process $\mathcal{F}_{[0,t]} \equiv X_{[0,t]}$ is equal to the probability of a random variable X_u at time $u \geq t$ (tomorrow) conditional only on the value of a random variable at time t (today). In other words, the history (sample path) of the stochastic process $\mathcal{F}_{[0,t]}$ is of no importance in that the way this stochastic process evolved or the dynamics does not mean a thing in terms of the conditional probability of the process. This property is sometimes called a memoryless property.

Definition 4.5 Markov Processes (formal) Consider a continuous time nonanticipating stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ which takes values in a measurable space $(B, \mathcal{B})$. $(B, \mathcal{B})$ is called a state space of the process and the process is said to be B - valued. Then, the process $(X_{t\in[0,\infty)})$ is said to be a Markov process if it satisfies, for an increasing sequence of time $0 \leq t \leq u \leq v < \infty$ $0 < t \leq u < \infty$:

$$E[X_v \left| \mathcal{F}_t \right.] = E[X_v \left| X_t \right.],$$

with the transition function:

$$\mathbb{P}_{t,v}(x, B) = \int_{\mathbb{R}} \mathbb{P}_{t,u}(x, dy) \mathbb{P}_{u,v}(y, B).$$

Now we are ready to characterize a standard Brownian motion as a subclass of Markov processes.

Theorem 4.1 A standard Brownian motion process A standard Brownian motion $(B_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ satisfies the followings:

(1) It is a time homogeneous Markov process. In other words, for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ and for $\forall 0 \leq t \leq u < \infty$:

$$E[f(B_u) \left| \mathcal{F}_t \right.] = \mathbb{P}_{0,u-t} f(B_t) = \mathbb{P}_{u-t} f(B_t).$$

(2) Its transition function $\mathbb{P}_{u-t} \equiv \mathbb{P}_h$ is given by:

$$\mathbb{P}_h(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\}.$$

(3) $\mathbb{P}_h f(x) = \begin{cases} f(x) & \text{if} \quad t = 0 \\ \int_{-\infty}^{\infty} \mathbb{P}_h(x,y) f(y) dy & \text{if} \quad t > 0 \end{cases}.$

Proof

Markov property is a result of independent increments property of Brownian motion. Let $(B_{t\in[0,\infty)})$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. Consider an increasing sequence of time $0 < t_1 < t_2 < ... < t_n < t < u < \infty$ where t is the present. As a result of independent increments condition:

$$\begin{aligned} &\mathbb{P}(X_u - X_t \,|\, X_{t_1} - X_0, X_{t_2} - X_{t_1}, ..., X_t - X_{t_n}) \\ &= \frac{\mathbb{P}(X_u - X_t \cap X_{t_1} - X_0, X_{t_2} - X_{t_1}, ..., X_t - X_{t_n})}{\mathbb{P}(X_{t_1} - X_0, X_{t_2} - X_{t_1}, ..., X_t - X_{t_n})} \\ &= \frac{\mathbb{P}(X_u - X_t)\mathbb{P}(X_{t_1} - X_0, X_{t_2} - X_{t_1}, ..., X_t - X_{t_n})}{\mathbb{P}(X_{t_1} - X_0, X_{t_2} - X_{t_1}, ..., X_t - X_{t_n})} \\ &= \mathbb{P}(X_u - X_t), \end{aligned}$$

which means that there is no correlation (probabilistic dependence structure) on the increments among the past, the present, and the future.

Using the simple relationship $X_u \equiv (X_u - X_t) + X_t$ for an increasing sequence of time $0 < t_1 < t_2 < ... < t_n < t < u < \infty$:

$$\begin{aligned} \mathbb{P}(X_u \,|\, X_0, X_{t_1}, X_{t_2}, ..., X_{t_n}, X_t) &= \mathbb{P}((X_u - X_t) + X_t \,|\, X_0, X_{t_1}, X_{t_2}, ..., X_{t_n}, X_t) \\ &= \mathbb{P}(X_u \,|\, X_t), \end{aligned}$$

which holds because an increment $(X_u - X_t)$ is independent of X_t by definition and the value of X_t depends on its realization $X_t(\omega)$.

□

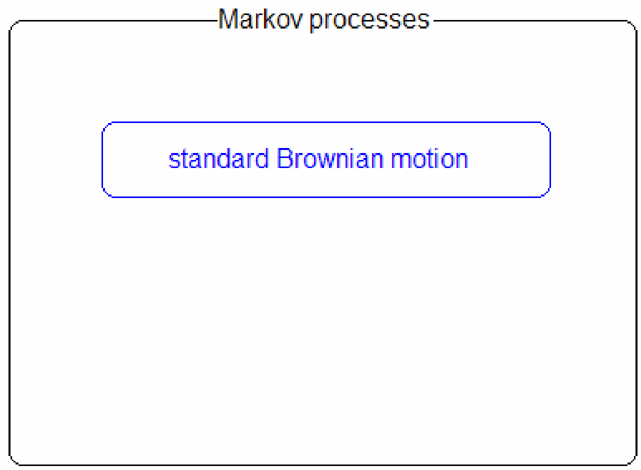


Figure 4.1 Brownian motion as a subclass of Markov processes

[5] Brownian Motion as a Subclass of Itô Diffusion Processes

Definition 5.1 Itô diffusion processes An Itô diffusion process is a real valued stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ whose dynamics (or motion) is governed by a stochastic differential equation of the form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

where $b(t, X_t) \in \mathbb{R}$ is called a drift and $\sigma(t, X_t)$ which is a nonnegative real valued constant is called a diffusion parameter. In the general case, $b(t, X_t)$ and $\sigma(t, X_t)$ are functions of both time and space. As usual, B_t stands for a standard Brownian motion. The solution of the above stochastic differential equation is given by:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

Definition 5.2 Time homogeneous (temporary homogeneous or stationary) Itô diffusion processes A time homogeneous Itô diffusion process is a real valued stochastic process $(X_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ whose dynamics (or motion) is governed by a stochastic differential equation of the form:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

where a drift $b(X_t) \in \mathbb{R}$ and a diffusion parameter $\sigma(X_t) \geq 0$ are independent of the time t and depend only on the space.

Theorem 5.1 Time homogeneous Itô diffusion processes are a subclass of time homogeneous Markov processes A time homogeneous Itô diffusion process $(X_{t\in[0,\infty)})$

defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is a time homogeneous Markov process. In other words, for any bounded Borel function $f:\mathbb{R}\to\mathbb{R}$ and for $\forall 0 \le t \le u < \infty$:

$$E[f(B_u)|\mathcal{F}_t] = \mathbb{P}_{0,u-t} f(B_t) = \mathbb{P}_{u-t} f(B_t),$$

where $\mathbb{P}_{u-t} \equiv \mathbb{P}_h$ is a time homogeneous transition function given by:

$$\mathbb{P}_h f(x) = \begin{cases} f(x) & \text{if} \quad t = 0 \\ \int_{-\infty}^{\infty} \mathbb{P}_h(x,y) f(y) dy & \text{if} \quad t > 0 \end{cases}.$$

Proof

Consult Oksendal (2003) pages 115-116.

Theorem 5.2 A standard Brownian motion A standard Brownian motion $(B_{t\in[0,\infty)})$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ is a time homogeneous Itô diffusion process $(X_{t\in[0,\infty)})$ (a time homogeneous Markov process) whose dynamics (or motion) is governed by a stochastic differential equation of the form:

$$dX_t = dB_t.$$

In other words, a standard Brownian motion $(B_{t\in[0,\infty)})$ is a time homogeneous Itô diffusion process with the zero drift $b(t, X_t) = 0$ and the unit diffusion parameter $\sigma(t, X_t) = 1$.

Proof

By the definition of a standard Brownian motion.

For more details about Itô diffusion processes, consult an excellent book Oksendal (2003) chapter 3 and 7.

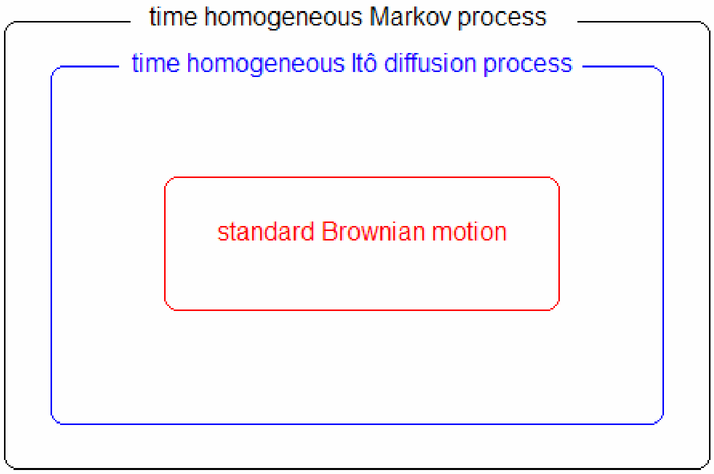


Figure 5.1 Brownian motion as a subclass of Markov processes

References

Karlin, S., and Taylor, H., 1975, A First Course in Stochastic Processes, Academic Press.

Karatzas, Ioannis., and Shreve, S. E., 1991, Brownian Motion and Stochastic Calculus, Springer-Verlag.

Oksendal, B., 2003, Stochastic Differential Equations: An Introduction with Applications, Springer.

Rogers, L.C.G., and Williams, D., 2000, Diffusions, Markov Processes and Martingales, Cambridge University Press.

Sato, K., 1999, Lévy process and Infinitely Divisible Distributions, Cambridge University Press.