Introduction to Black-Scholes Model

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Abstract

This paper presents everything you need to know about Black-Scholes model which is truly single most important revolutionary work in the history of quantitative finance. Although BS model has its flaws such as the normally distributed (i.e. zero skewness and zero excess kurtosis) log return density and the assumption of constant volatility across strike prices and the time to maturity, it outperforms more (so-called) advanced models in numerous cases.

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[1] Standard Brownian Motion: Building Block of BS Model

A standard Brownian motion $(B_{t\in[0,\infty)})$ is a real valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ satisfying:

(1) Its increments are independent. In other words, for $0 \leq t_1 < t_2 < ... < t_n < \infty$:

$$\mathbb{P}(B_{t_0} \cap B_{t_1} - B_{t_0} \cap B_{t_2} - B_{t_1} \cap ... \cap B_{t_n} - B_{t_{n-1}})$$
$$= \mathbb{P}(B_{t_0})\mathbb{P}(B_{t_1} - B_{t_0})\mathbb{P}(B_{t_2} - B_{t_1})...\mathbb{P}(B_{t_n} - B_{t_{n-1}}).$$

(2) Its increments are stationary (time homogeneous): i.e. for $h \geq 0$, $B_{t+h} - B_t$ has the same distribution as B_h. In other words, the distribution of increments does not depend on t.
(3) $\mathbb{P}(B_0 = 0) = 1$. The process starts from 0 almost surely (with probability 1).
(4) $B_t \sim Normal(0,t)$. Its increments follow a Gaussian distribution with the mean 0 and the variance t.

It turns out that a standard Brownian motion $(B_{t\in[0,\infty)})$ satisfies the following conditions:

(1) The process is stochastically continuous: $\forall \varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.
(2) Its sample path (trajectory) is continuous in t (i.e. continuous $\in$ rcll) almost surely.

For more details about Brmwoan motion, consult Matsuda (2005) "Introduction to Brownian Motion".

[2] Black-Scholes' Distributional Assumptions on a Stock Price

In traditional finance literature almost every financial asset price (stocks, currencies, interest rates) is assumed to follow some variations of Brownian motion with drift process. BS (Black-Scholes) models a stock price increment process in an infinitesimal time interval dt as a log-normal random walk process:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1}$$

where the drift is μS_t which is a constant expected return on a stock μ proportional to a stock price S_t and the volatility is σS_t which is a constant stock price volatility σ proportional to a stock price S_t. The reason why the process (1) is called a log-normal random walk process will be explained very soon. Alternatively, we can state that BS models a percentage change in a stock price process in an infinitesimal time interval dt as a Brownian motion with drift process:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t , \tag{2}$$

$$\mathbb{P}(\frac{dS_t}{S_t}) = \frac{1}{\sqrt{2\pi\sigma^2 dt}} \exp[-\frac{(dS_t / S_t - \mu dt)^2}{2\sigma^2 dt}] .$$

Let S be a random variable whose dynamics is given by an Ito process:

$$dS = a(S,t)dt + b(S,t)dB ,$$

and V be a function dependent on a random variable S and time t. The dynamics of $V(S,t)$ is given by an Ito formula:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} dt , \tag{3}$$

or in terms of a standard Brownian motion process B:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}\left(adt + bdB \right) + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} dt ,$$

$$dV = \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial S} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} \right) dt + b\frac{\partial V}{\partial S} dB . \tag{4}$$

Dynamics of a log stock price process $\ln S_t$ can be obtained by applying (4) to (1) as:

$$d\ln S_t = \left(\frac{\partial \ln S_t}{\partial t} + \mu S_t \frac{\partial \ln S_t}{\partial S_t} + \frac{1}{2}\sigma^2 {S_t}^2 \frac{\partial^2 \ln S_t}{\partial {S_t}^2} \right) dt + \sigma S_t \frac{\partial \ln S_t}{\partial S_t} dB_t .$$

Substituting $\frac{\partial \ln S_t}{\partial t} = 0$, $\frac{\partial \ln S_t}{\partial S_t} = \frac{1}{S_t}$, and $\frac{\partial^2 \ln S_t}{\partial {S_t}^2} = -\frac{1}{{S_t}^2}$ yields:

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dB_t , \tag{5}$$

or:

$$\ln S_t - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2 \right)(t-0) + \sigma(B_t - B_0)$$

$$\ln S_t = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B_t . \tag{6}$$

The equation (6) means that BS models a log stock price $\ln S_t$ as a Brownian motion with drift process whose probability density is given by a normal density:

$$\mathbb{P}(\ln S_t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp[-\frac{\left\{ \ln S_t - \left(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t \right) \right\}^2}{2\sigma^2 t}]. \tag{7}$$

Alternatively, the equation (6) means that BS models a log return $\ln\left(S_t / S_0\right)$ as a Brownian motion with drift process whose probability density is given by a normal density:

$$\ln\left(S_t / S_0\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t,$$

$$\mathbb{P}\left(\ln\left(S_t / S_0\right)\right) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp[-\frac{\left\{ \ln\left(S_t / S_0\right) - \left(\mu - \frac{1}{2}\sigma^2\right)t \right\}^2}{2\sigma^2 t}]. \tag{8}$$

An example of BS normal log return $\ln\left(S_t / S_0\right)$ density of (8) is illustrated in Figure 1. Of course, BS log return density is symmetric (i.e. zero skewness) and have zero excess kurtosis because it is a normal density.

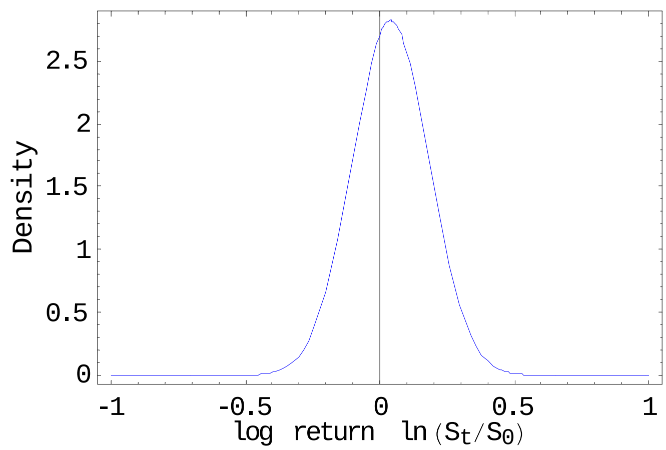

Figure 1: An Example of BS normal log return ln(S_t/S_0) Density. Parameters and variables fixed are μ = 0.1, σ = 0.2, and t = 0.5.

Let y be a random variable. If the log of y is normally distributed with mean a and variance b^2 such that $\ln y \sim N(a, b^2)$, then y is a log-normal random variable whose density is a two parameter family (a, b):

$$y \sim Lognormal(e^{a+\frac{1}{2}b^2}, e^{2a+b^2}(e^{b^2} - 1)),$$

$$\mathbb{P}(y) = \frac{1}{y\sqrt{2\pi b^2}} \exp[-\frac{\{\ln y - a\}^2}{2b^2}].$$

From the equation (6), we can state that BS models a stock price S_t as a log-normally distributed random variable whose density is given by:

$$\mathbb{P}(S_t) = \frac{1}{S_t\sqrt{2\pi\sigma^2 t}} \exp[-\frac{\left\{\ln S_t - \left(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t\right)\right\}^2}{2\sigma^2 t}]. \quad (9)$$

Its annualized moments are calculated as:

$$Mean[S_t] = S_0 e^{\mu},$$
$$Variance[S_t] = S_0^{\ 2}\left(e^{\sigma^2} - 1\right)e^{2\mu},$$
$$Skewness[S_t] = \left(e^{\sigma^2} + 2\right)\sqrt{e^{\sigma^2} - 1},$$
$$Excess\ Kurtosis[S_t] = -6 + 3e^{2\sigma^2} + 2e^{3\sigma^2} + e^{4\sigma^2}.$$

An example of BS log-normal stock price density of (9) is illustrated in Figure 2. Notice that BS log-normal stock price density is positively skewed.

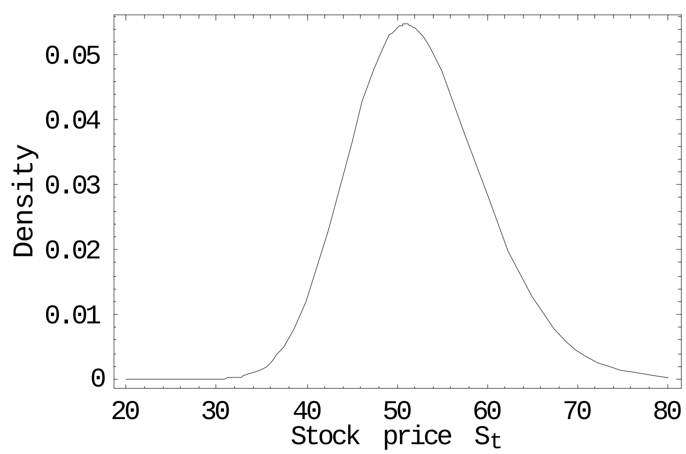

Figure 2: An Example of BS Log-Normal Density of a Stock Price. Parameters and variables fixed are S_0 = 50, μ = 0.1, σ = 0.2, and t = 0.5.

Table 1
Annualized Moments of BS Log-Normal Density of A Stock Price in Figure 2

Mean	Standard Deviation	Skewness	Excess Kurtosis
55.2585	11.1632	0.614295	0.678366

From the equation (6), we can obtain an integral version equivalent of (1):

$$\exp[\ln S_t] = \exp[\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t]$$
$$S_t = \exp[\ln S_0]\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]$$
$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]. \quad (10)$$

Equation (10) means that BS models a stock price dynamics as a geometric (i.e. exponential) process with the growth rate given by a Brownian motion with drift process:

$$\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t .$$

[3] Traditional Black-Scholes Option Pricing: PDE Approach by Hedging

Consider a portfolio P of the one long option position $V(S,t)$ on the underlying stock S written at time t and a short position of the underlying stock in quantity Δ to derive option pricing function.

$$P_t = V(S_t, t) - \Delta S_t . \quad (11)$$

Portfolio value changes in a very short period of time dt by:

$$dP_t = dV(S_t, t) - \Delta dS_t . \quad (12)$$

Stock price dynamics is given by a log-normal random walk process of the equation (7.1):

$$dS_t = \mu S_t dt + \sigma S_t dB_t . \quad (13)$$

Option price dynamics is given by applying Ito formula of the equation (3):

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_t}dS_t + \frac{1}{2}\sigma^2 S_t^{\,2}\frac{\partial^2 V}{\partial S_t^{\,2}}dt . \quad (14)$$

Now the change in the portfolio value can be expressed as by substituting (13) and (14) into (12):

$$dP_t = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_t} dS_t + \frac{1}{2}\sigma^2 S_t^{\ 2} \frac{\partial^2 V}{\partial S_t^{\ 2}} dt - \Delta dS_t . \tag{15}$$

Setting $\Delta = \partial V / \partial S_t$ (i.e. delta hedging) makes the portfolio completely risk-free (i.e. the randomness dS_t has been eliminated) and the portfolio value dynamics of the equation (15) simplifies to:

$$dP_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^{\ 2} \frac{\partial^2 V}{\partial S_t^{\ 2}} \right) dt . \tag{16}$$

Since this portfolio is perfectly risk-free, assuming the absence of arbitrage opportunities the portfolio is expected to grow at the risk-free interest rate r :

$$E[dP_t] = rP_t dt . \tag{17}$$

After substitution of (11) and (16) into (17) by setting $\Delta = \partial V / \partial S_t$, we obtain:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^{\ 2} \frac{\partial^2 V}{\partial S_t^{\ 2}} \right) dt = r \left(V - \frac{\partial V}{\partial S_t} S_t \right) dt .$$

After rearrangement, Black-Scholes PDE is obtained:

$$\frac{\partial V(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^{\ 2} \frac{\partial^2 V(S_t, t)}{\partial S_t^{\ 2}} + rS_t \frac{\partial V(S_t, t)}{\partial S_t} - rV(S_t, t) = 0 . \tag{18}$$

BS PDE is categorized as a linear second-order parabolic PDE. The equation (18) is a linear PDE because coefficients of the partial derivatives of $V(S_t, t)$ (i.e. $\sigma^2 S_t^{\ 2} / 2$ and rS_t) are not functions of $V(S_t, t)$ itself. The equation (18) is a second-order PDE because it involves the second-order partial derivative $\partial^2 V(S_t, t) / \partial S_t^{\ 2}$. Generally speaking, a PDE of the form:

$$a + b\frac{\partial V}{\partial t} + c\frac{\partial V}{\partial S} + d\frac{\partial^2 V}{\partial S^2} + e\frac{\partial^2 V}{\partial t^2} + g\frac{\partial^2 V}{\partial t \partial S} = 0 ,$$

is said to be a parabolic type if:

$$g - 4de = 0 . \tag{19}$$

The equation (18) is a parabolic PDE because it has $g = 0$ and $e = 0$ which satisfies the condition (19).

BS solves PDE of (18) with boundary conditions:

$$\max\left(S_T - K, 0\right) \text{ for a plain vanilla call,}$$
$$\max\left(K - S_T, 0\right) \text{ for a plain vanilla put,}$$

and obtains closed-form solutions of call and put pricing functions. Exact derivation of closed-form solutions by solving BS PDE is omitted here (i.e. the original BS approach). Instead we will provide the exact derivation by a martingale asset pricing approach (this is much simpler) in the next section.

[4] Traditional Black-Scholes Option Pricing: Martingale Pricing Approach

Let $\{B_t; 0 \le t \le T\}$ be a standard Brownian motion process on a space $(\Omega, \mathcal{F}, \mathbb{P})$. Under actual probability measure $\mathbb{P}$, the dynamics of BS stock price process is given by equation (9) in the integral form (i.e. which is a geometric Brownian motion process):

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]. \tag{20}$$

BS model is an example of a complete model because there is only one equivalent martingale risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ under which the discounted asset price process $\{e^{-rt}S_t; 0 \le t \le T\}$ becomes a martingale. BS finds the equivalent martingale risk-neutral measure $\mathbb{Q}_{BS} \sim \mathbb{P}$ by changing the drift of the Brownian motion process while keeping the volatility parameter σ unchanged:

$$S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t^{\mathbb{Q}_{BS}}\right]. \tag{21}$$

Note that $B_t^{\mathbb{Q}_{BS}}$ is a standard Brownian motion process on $(\Omega, \mathcal{F}, \mathbb{Q}_{BS})$ and the discounted stock price process $\{e^{-rt}S_t; 0 \le t \le T\}$ is a martingale under $\mathbb{Q}_{BS}$ and with respect to the filtration $\{\mathcal{F}_t; 0 \le t \le T\}$. Then, a plain vanilla call option price $C(t, S_t)$ which has a terminal payoff function $\max\left(S_T - K, 0\right)$ is calculated as:

$$C(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}_{BS}}\left[\max\left(S_T - K, 0\right) \middle| \mathcal{F}_t\right]. \tag{22}$$

Let $\mathbb{Q}(S_T)$ (drop the subscript BS for simplicity) be a probability density function of S_T in a risk-neutral world. From the equation (9), a terminal stock price S_T is a log-normal random variable with its density of the form:

$$\mathbb{Q}\left(S_T\right)=\frac{1}{S_T\sqrt{2\pi\sigma^2\tau}}\exp[-\frac{\left\{\ln S_T-\left(\ln S_t+(r-\frac{1}{2}\sigma^2)\tau\right)\right\}^2}{2\sigma^2\tau}]. \tag{23}$$

Using (23), the expectation term in (22) can be rewritten as:

$$E^{\mathbb{Q}}\left[\max\left(S_T-K,0\right)\middle|\mathcal{F}_t\right]=\int_K^\infty\left(S_T-K\right)\mathbb{Q}\left(S_T\middle|\mathcal{F}_t\right)dS_T+\int_0^K(0)\mathbb{Q}\left(S_T\middle|\mathcal{F}_t\right)dS_T$$
$$E^{\mathbb{Q}}\left[\max\left(S_T-K,0\right)\middle|\mathcal{F}_t\right]=\int_K^\infty\left(S_T-K\right)\mathbb{Q}\left(S_T\middle|\mathcal{F}_t\right)dS_T\,.$$

Using this, we can rewrite (22) by setting $\tau\equiv T-t$ as:

$$C(\tau,S_t)=e^{-r\tau}\int_K^\infty\left(S_T-K\right)\mathbb{Q}\left(S_T\middle|\mathcal{F}_t\right)dS_T\,. \tag{24}$$

Since S_T is a log-normal random variable with its density given by the equation (23):

$$\ln S_T\left|\mathcal{F}_t\right.\sim Normal\left(m\equiv\ln S_t+(r-\frac{1}{2}\sigma^2)\tau,\sigma^2\tau\right). \tag{25}$$

For the notational simplicity, let $\ln S_T\left|\mathcal{F}_t\right.\equiv\ln S_T$. We use a change of variable technique from a log-normal random variable S_T to a standard normal random variable Z through:

$$Z\equiv\frac{\ln S_T-\left\{\ln S_t+(r-\frac{1}{2}\sigma^2)\tau\right\}}{\sigma\sqrt{\tau}}\equiv\frac{\ln S_T-m}{\sigma\sqrt{\tau}}\sim Normal\left(0,1\right), \tag{26}$$

with:

$$\mathbb{Z}\left(Z\right)=\frac{1}{\sqrt{2\pi}}\exp[-\frac{Z^2}{2}].$$

From (26):

$$S_T=\exp\left(Z\sigma\sqrt{\tau}+m\right). \tag{27}$$

We can rewrite (24) as:

$$C(\tau,S_t)=e^{-r\tau}\int_{(\ln K-m)/\sigma\sqrt{\tau}}^\infty\left(\exp\left(Z\sigma\sqrt{\tau}+m\right)-K\right)\mathbb{Z}\left(Z\right)dZ\,,$$

and we express this with more compact form as:

$$C(\tau, S_t) = C_1 - C_2 , \tag{28}$$

where $C_1 = e^{-r\tau} \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \exp\left(Z\sigma\sqrt{\tau} + m\right) \mathbb{Z}(Z)\, dZ$ and $C_2 = Ke^{-r\tau} \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \mathbb{Z}(Z)\, dZ$.

Consider C_1 :

$$C_1 = e^{-r\tau} \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \exp\left(Z\sigma\sqrt{\tau}\right) \exp(m) \mathbb{Z}(Z)\, dZ$$

$$C_1 = \exp(-r\tau) \exp\left(\ln S_t + (r - \frac{1}{2}\sigma^2)\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \exp\left(Z\sigma\sqrt{\tau}\right) \mathbb{Z}(Z)\, dZ$$

$$C_1 = \exp\left(\ln S_t - \frac{1}{2}\sigma^2\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \exp\left(Z\sigma\sqrt{\tau}\right) \mathbb{Z}(Z)\, dZ$$

$$C_1 = \exp\left(\ln S_t - \frac{1}{2}\sigma^2\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \exp\left(Z\sigma\sqrt{\tau}\right) \frac{1}{\sqrt{2\pi}} \exp[-\frac{Z^2}{2}] dZ$$

$$C_1 = \exp\left(\ln S_t - \frac{1}{2}\sigma^2\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{Z^2 - 2Z\sigma\sqrt{\tau}}{2}] dZ$$

$$C_1 = \exp\left(\ln S_t - \frac{1}{2}\sigma^2\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{\left(Z - \sigma\sqrt{\tau}\right)^2 - \sigma^2\tau}{2}] dZ$$

$$C_1 = \exp\left(\ln S_t - \frac{1}{2}\sigma^2\tau\right) \exp\left(\frac{1}{2}\sigma^2\tau\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{\left(Z - \sigma\sqrt{\tau}\right)^2}{2}] dZ$$

$$C_1 = \exp\left(\ln S_t\right) \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{\left(Z - \sigma\sqrt{\tau}\right)^2}{2}] dZ$$

$$C_1 = S_t \int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{\left(Z - \sigma\sqrt{\tau}\right)^2}{2}] dZ . \tag{29}$$

Use the following relationship:

$$\int_a^b \frac{1}{\sqrt{2\pi}} \exp[-\frac{\left(Z - c\right)^2}{2}] dZ = \int_{a-c}^{b-c} \frac{1}{\sqrt{2\pi}} \exp[-\frac{Z^2}{2}] dZ .$$

Equation (29) can be rewritten as:

$$C_1 = S_t \int_{(\ln K - m)/\sigma\sqrt{\tau} - \sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{Z^2}{2}] dZ . \tag{30}$$

Let $N(\ )$ be the standard normal cumulative density function. Using the symmetry of a normal density, (30) can be rewritten as:

$$C_1 = S_t \int_{-\infty}^{-(\ln K - m)/\sigma\sqrt{\tau}+\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} \exp[-\frac{Z^2}{2}]dZ$$
$$C_1 = S_t N\left(\frac{-\ln K + m}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\right). \tag{31}$$

From (25), substitute for m. The equation (31) becomes:

$$C_1 = S_t N\left(\frac{-\ln K + \ln S_t + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\right)$$
$$C_1 = S_t N\left(\frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{1}{2}\sigma^2)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) = S_t N\left(\frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \tag{32}$$

Next, consider C_2 in (28):

$$C_2 = Ke^{-r\tau}\int_{(\ln K - m)/\sigma\sqrt{\tau}}^{\infty} \mathbb{Z}(Z)\,dZ = Ke^{-r\tau}\int_{-\infty}^{-(\ln K - m)/\sigma\sqrt{\tau}} \mathbb{Z}(Z)\,dZ$$
$$C_2 = Ke^{-r\tau} N\left(\frac{-\ln K + m}{\sigma\sqrt{\tau}}\right). \tag{33}$$

From (25), substitute for m. The equation (33) becomes:

$$C_2 = Ke^{-r\tau} N\left(\frac{-\ln K + \ln S_t + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) = Ke^{-r\tau} N\left(\frac{\ln \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \tag{34}$$

Substitute (32) and (34) into (28) and we obtain BS plain vanilla call option pricing formula:

$$C(\tau, S_t) = S_t N(d_1) - Ke^{-r\tau} N(d_2), \tag{35}$$

where $d_1 = \dfrac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and $d_2 = \dfrac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}$.

Following the similar method, BS plain vanilla put option pricing formula can be obtained as:

$$P(\tau, S_t) = Ke^{-r\tau} N\left(-d_2\right) - S_t N\left(-d_1\right). \tag{36}$$

We conclude that both PDE approach and martingale approach give the same result. This is because in both approaches we move from a historical probability measure $\mathbb{P}$ to a risk-neutral probability measure $\mathbb{Q}$. This is very obvious for martingale method. But in PDE approach because the source of randomness can be completely eliminated by forming a portfolio of options and underlying stocks, this portfolio grows at a rate equal to the risk-free interest rate. Thus, we switch to a measure $\mathbb{Q}$. For more details, we recommend Neftci (2000) pages 280-282 and 358-365.

[5] Alternative Interpretation of Black-Scholes Formula: A Single Integration Problem

Under an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ under which the discounted asset price process $\{e^{-rt} S_t ; 0 \le t \le T\}$ becomes a martingale, a plain vanilla call and put option price which has a terminal payoff function $\max\left(S_T - K, 0\right)$ and $\max\left(K - S_T, 0\right)$ are calculated as:

$$C(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right) \middle| \mathcal{F}_t\right], \tag{37}$$
$$P(t, S_t) = e^{-r(T-t)} E^{\mathbb{Q}}\left[\max\left(K - S_T, 0\right) \middle| \mathcal{F}_t\right]. \tag{38}$$

Note that an expectation operator $E[\;]$ is under a probability measure $\mathbb{Q}$ and with respect to the filtration $\mathcal{F}_t$. Let $\mathbb{Q}(S_T \,|\mathcal{F}_t)$ be a conditional probability density function of a terminal stock price S_T. For the notational simplicity we use $\mathbb{Q}(S_T \,|\mathcal{F}_t) \equiv \mathbb{Q}(S_T)$ and $\tau \equiv T - t$. The expected terminal payoffs in the equations (37) and (38) can be rewritten as:

$$E^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right) \middle| \mathcal{F}_t\right] = \int_K^\infty \left(S_T - K\right) \mathbb{Q}\left(S_T\right) dS_T,$$
$$E^{\mathbb{Q}}\left[\max\left(K - S_T, 0\right) \middle| \mathcal{F}_t\right] = \int_0^K \left(K - S_T\right) \mathbb{Q}\left(S_T\right) dS_T.$$

Using these, we can rewrite (37) and (38) as:

$$C(\tau, S_t) = e^{-r\tau} \int_K^\infty \left(S_T - K\right) \mathbb{Q}\left(S_T\right) dS_T, \tag{39}$$

$$P(\tau, S_t) = e^{-r\tau} \int_0^K (K - S_T) \mathbb{Q}(S_T) dS_T . \tag{40}$$

BS assumes that a terminal stock price S_T is a log-normal random variable with its density of the form:

$$\mathbb{Q}(S_T) = \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp[-\frac{\left\{ \ln S_T - \left(\ln S_t + (r - \frac{1}{2}\sigma^2)\tau \right) \right\}^2}{2\sigma^2\tau}].$$

Therefore, BS option pricing formula comes down to a very simple single integration problem:

$$C(\tau, S_t) = e^{-r\tau} \int_K^\infty (S_T - K) \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp[-\frac{\left\{ \ln S_T - \left(\ln S_t + (r - \frac{1}{2}\sigma^2)\tau \right) \right\}^2}{2\sigma^2\tau}] dS_T , \tag{41}$$

$$P(\tau, S_t) = e^{-r\tau} \int_0^K (K - S_T) \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp[-\frac{\left\{ \ln S_T - \left(\ln S_t + (r - \frac{1}{2}\sigma^2)\tau \right) \right\}^2}{2\sigma^2\tau}] dS_T . \tag{42}$$

This implies that as far as a risk-neutral conditional density of the terminal stock price $\mathbb{Q}(S_T | \mathcal{F}_t)$ is known, plain vanilla option pricing reduces to a simple integration problem.

[6] Black-Scholes Model as an Exponential Lévy Model

The equation (10) tells us that BS models a stock price process as an exponential Brownian motion with drift process:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right],$$

which means:

$$S_t = S_0 \, e^{L_t} ,$$

where the stock price process $\{S_t : 0 \le t \le T\}$ is modeled as an exponential of a Lévy process $\{L_t ; 0 \le t \le T\}$. Black and Scholes' choice of the Lévy process is a Brownian motion with drift (continuous diffusion process):

$$L_t \equiv \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B_t . \tag{43}$$

The fact that an stock price S_t is modeled as an exponential of Lévy process L_t means that its log-return $\ln(\frac{S_t}{S_0})$ is modeled as a Lévy process such that:

$$\ln(\frac{S_t}{S_0}) = L_t = \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B_t .$$

BS model can be categorized as the only continuous exponential Lévy model apparently because a Brownian motion with drift process is the only continuous (i.e. no jumps) Lévy process. This indicats that the Lévy measure of a Brownian motion with drift process is zero:

$$\ell(dx) = 0 ,$$

and obviously its arrival rate of jumps is zero:

$$\int \ell(dx) = 0 .$$

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