Inverse Gaussian Distribution

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Abstract

This paper presents the basic knowledge of the inverse Gaussian distribution.

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[1] Inverse Gaussian Distribution: JKB (1994) Parameterization

There are many different parameterizations of the inverse Gaussian distribution which can be really confusing to beginners. In this section, basic properties of the inverse Gaussian distribution is presented following Johnson, Kotz, and Balakrishnan (1994)'s parameterization of the equation (15.4a).

The probability density function of the inverse Gaussian distribution is a two parameter family:

$$IG(z;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi}}z^{-3/2}\exp\left\{-\frac{\lambda}{2\mu^2 z}(z-\mu)^2\right\}1_{x>0}\,, \tag{1}$$

where $\mu\in\mathbb{R}^+$ and $\lambda\in\mathbb{R}^+$.

By Fourier transforming the IG probability density (1), its characteristic function $\phi_Z(\omega)$ is calculated as:

$$\phi_Z(\omega)\equiv\mathcal{F}_Z[IG(z;\mu,\lambda)](\omega)\equiv\int_{-\infty}^{\infty}e^{i\omega z}IG(z)dz$$
$$\phi_Z(\omega)=\int_0^{\infty}e^{i\omega z}IG(z)dz=\exp\left(\frac{\lambda}{\mu}-\sqrt{\lambda}\sqrt{\frac{\lambda}{\mu^2}-2i\omega}\right). \tag{2}$$

Simplifying the equation (2) yields:

$$\phi_Z(\omega)=\exp\left(\frac{\lambda}{\mu}-\sqrt{\frac{\lambda^2}{\mu^2}-2\lambda i\omega}\right)=\exp\left\{\frac{\lambda}{\mu}\left(1-\frac{\mu}{\lambda}\sqrt{\frac{\lambda^2}{\mu^2}-2\lambda i\omega}\right)\right\}$$
$$\phi_Z(\omega)=\exp\left\{\frac{\lambda}{\mu}\left(1-\sqrt{\frac{\mu^2}{\lambda^2}\frac{\lambda^2}{\mu^2}-\frac{\mu^2}{\lambda^2}2\lambda i\omega}\right)\right\}$$
$$\phi_Z(\omega)=\exp\left\{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 i\omega}{\lambda}}\right)\right\}. \tag{3}$$

The characteristic exponent (i.e. cumulant generating function) $\psi_Z(\omega)$ of the IG distribution is:

$$\psi_Z(\omega)\equiv\ln\phi_Z(\omega)=\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 i\omega}{\lambda}}\right). \tag{4}$$

Using (4), the first four cumulants defined by $cumulant_n(Z) \equiv \frac{1}{i^n} \frac{\partial^n \psi_Z(\omega)}{\partial \omega^n}\Big|_{\omega=0}$ are calculated as the follows:

$$
\begin{aligned}
&cumulant_1 = \mu , \\
&cumulant_2 = \mu^3 / \lambda , \\
&cumulant_3 = 3\mu^5 / \lambda^2 , \\
&cumulant_4 = 15\mu^7 / \lambda^3 .
\end{aligned}
$$

Using the above cumulants, the mean, variance, skewness, and excess kurtosis of the IG random variable Z are obtained as (consult Table 4.1 of Matsuda (2004)):

$$
\begin{aligned}
&E[Z] = \mu , \\
&Variance[Z] = \mu^3 / \lambda , \\
&Skewness[Z] = 3\sqrt{\mu / \lambda} , \\
&Excess\ Kurtosis[Z] = 15\mu / \lambda .
\end{aligned}
\tag{5}
$$

The moment generating function $M_Z(\omega)$ of the IG distribution can be expressed as:

$$
\begin{aligned}
&M_Z(\omega) \equiv \int_{-\infty}^{\infty} e^{\omega z} IG(z; \mu, \lambda) dz = \int_0^{\infty} e^{\omega z} IG(z) dz \\
&M_Z(\omega) = \exp\left(\xi_Z(\omega)\right),
\end{aligned}
$$

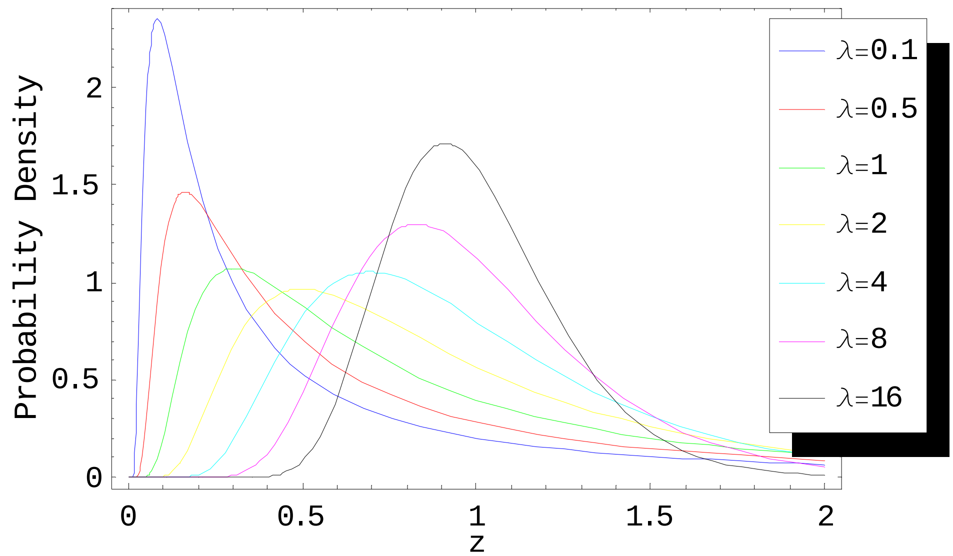
where the Laplace exponent $\xi_Z(\omega)$ is given by:

$$
\xi_Z(\omega) = \frac{\lambda}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2 \omega}{\lambda}}\right). \tag{6}
$$

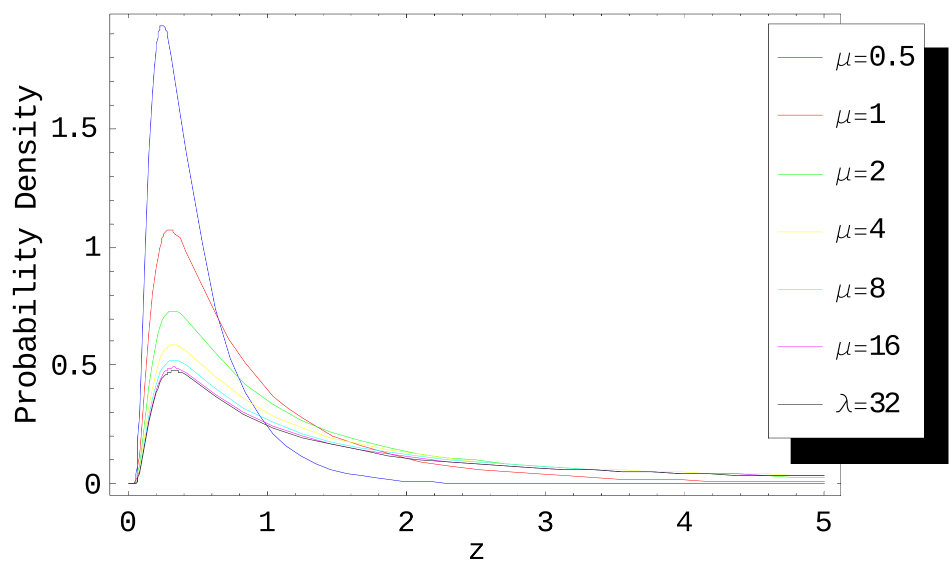
Using the moment generating function $M_Z(\omega)$ with (6), first four raw moments (i.e. $r_n \equiv \frac{\partial^n M_Z(\omega)}{\partial \omega^n}\Big|_{\omega=0}$) of the IG distribution are computed as:

$$
\begin{aligned}
&r_1 \equiv E[Z] = \mu , \\
&r_2 \equiv E[Z^2] = \mu^2 + \frac{\mu^3}{\lambda} , \\
&r_3 \equiv E[Z^3] = \mu^3 + \frac{3\mu^4}{\lambda} + \frac{3\mu^5}{\lambda^2} , \\
&r_4 \equiv E[Z^4] = \mu^4 + \frac{6\mu^5}{\lambda} + \frac{15\mu^6}{\lambda^2} + \frac{15\mu^7}{\lambda^3} .
\end{aligned}
$$

Note that the form of centered moments of (5) tells us that the IG probability density is always positively skewed and the excess kurtosis is always positive. Figure 1 illustrates the shape of the IG distribution with varying parameters. In Panel A, as λ increases, its variance, skewness, and excess kurtosis decreases. In Panel B, as μ rises holding λ constant, all moments rise.



A) μ = 1 and varying λ



B) Varying μ and λ = 1

Figure 1 Plot of IG probability density

Probably, the most important property of the IG distribution is its infinite divisibility. Let Z be an IG random variable with $\mu \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$. Then, there exist n pieces of $i.i.d.$ random variable $Z_1, Z_2, ..., Z_n$ each from the IG distribution with $\mu / n \in \mathbb{R}^+$ and $\lambda / n \in \mathbb{R}^+$ such that:

$$Z \underset{=}{d} Z_1 + Z_2 + ... + Z_n ,$$

which is the definition 3.3 of Matsuda (2005). This indicates that the IG distribution generates a class of increasing Lévy processes (subordinators).

[2] Inverse Gaussian Distribution: Barndorff-Nielsen (1998) Parameterization

In this section, basic properties of the inverse Gaussian distribution is presented following Barndorff-Nielsen (1998)'s parameterization.

Reparameterize the IG probability density (1) using $\mu = \delta / \gamma$ and $\lambda = \delta^2$:

$$IG(z;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}} z^{-3/2} \exp\left\{ -\frac{\lambda}{2\mu^2 z}(z-\mu)^2 \right\} 1_{x>0}$$

$$IG(z;\delta,\gamma) = \sqrt{\frac{\delta^2}{2\pi}} z^{-3/2} \exp\left\{ -\frac{\delta^2}{2(\delta/\gamma)^2 z}\left(z - \frac{\delta}{\gamma} \right)^2 \right\} 1_{x>0}$$

$$IG(z;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} z^{-3/2} \exp\left\{ -\frac{\gamma^2}{2z}\left(z^2 - 2z\frac{\delta}{\gamma} + \frac{\delta^2}{\gamma^2} \right) \right\} 1_{x>0}$$

$$IG(z;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} z^{-3/2} \exp\left\{ -\frac{\gamma^2 z}{2} + \delta\gamma - \frac{\delta^2}{2z} \right\} 1_{x>0}$$

$$IG(z;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} z^{-3/2} \exp\left\{ \delta\gamma - \frac{\left(\delta^2 z^{-1} + \gamma^2 z \right)}{2} \right\} 1_{x>0} , \qquad (7)$$

where $\delta \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$.

By Fourier transforming the IG probability density (7), its characteristic function $\phi_Z(\omega)$ is calculated as:

$$\phi_Z(\omega) \equiv \mathcal{F}_Z[IG(z;\delta,\gamma)](\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega z} IG(z) dz$$

$$\phi_Z(\omega) = \int_0^{\infty} e^{i\omega z} IG(z) dz = \exp\left(\delta\gamma - \delta\sqrt{\gamma^2 - 2i\omega} \right). \qquad (8)$$

The characteristic exponent (i.e. cumulant generating function) $\psi_Z(\omega)$ of the IG distribution is:

$$\psi_Z(\omega) \equiv \ln \phi_Z(\omega) = \delta\gamma - \delta\sqrt{\gamma^2 - 2i\omega} . \tag{9}$$

Using (9), the first four cumulants defined by $cumulant_n(Z) \equiv \frac{1}{i^n} \frac{\partial^n \psi_Z(\omega)}{\partial \omega^n}\Big|_{\omega=0}$ are calculated as the follows:

$$\begin{aligned} cumulant_1 &= \delta / \gamma , \\ cumulant_2 &= \delta / \gamma^3 , \\ cumulant_3 &= 3\delta / \gamma^5 , \\ cumulant_4 &= 15\delta / \gamma^7 . \end{aligned}$$

Using the above cumulants, the mean, variance, skewness, and excess kurtosis of the IG random variable Z are obtained as (consult Table 4.1 of Matsuda (2004)):

$$\begin{aligned} E[Z] &= \delta / \gamma , \\ Variance[Z] &= \delta / \gamma^3 , \\ Skewness[Z] &= \frac{3}{\sqrt{\delta\gamma}} , \\ Excess\ Kurtosis[Z] &= \frac{15}{\delta\gamma} . \end{aligned} \tag{10}$$

The moment generating function $M_Z(\omega)$ of the IG distribution can be expressed as:

$$M_Z(\omega) \equiv \int_{-\infty}^{\infty} e^{\omega z} IG(z; \delta, \gamma) dz = \int_0^{\infty} e^{\omega z} IG(z) dz$$
$$M_Z(\omega) = \exp\left(\xi_Z(\omega)\right),$$

where the Laplace exponent $\xi_Z(\omega)$ is given by:

$$\xi_Z(\omega) = \delta\gamma - \delta\sqrt{\gamma^2 - 2\omega} . \tag{11}$$

Using the moment generating function $M_Z(\omega)$ with (11), first four raw moments (i.e. $r_n \equiv \frac{\partial^n M_Z(\omega)}{\partial \omega^n}\Big|_{\omega=0}$) of the IG distribution are computed as:

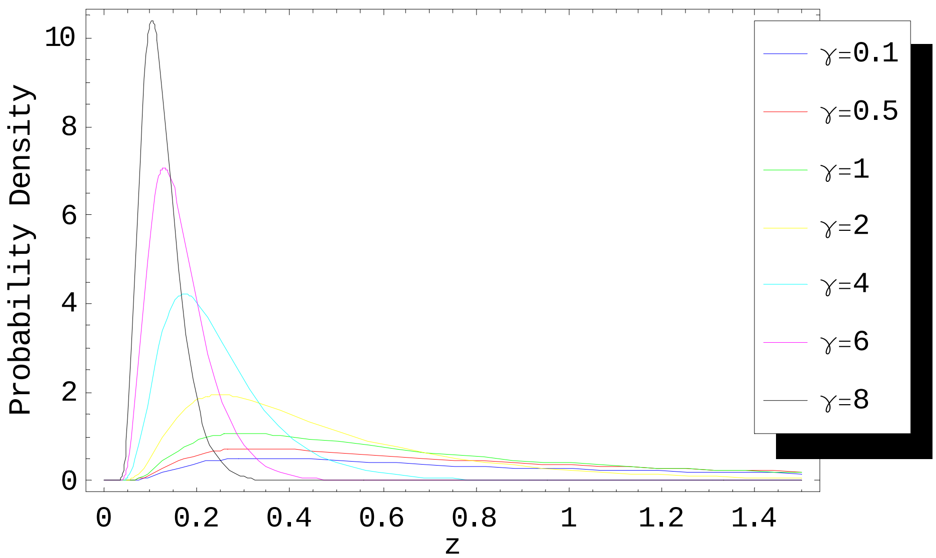
$$r_1 \equiv E[Z] = \delta / \gamma ,$$

$$r_2 \equiv E[Z^2] = \frac{\delta(1+\delta\gamma)}{\gamma^3},$$

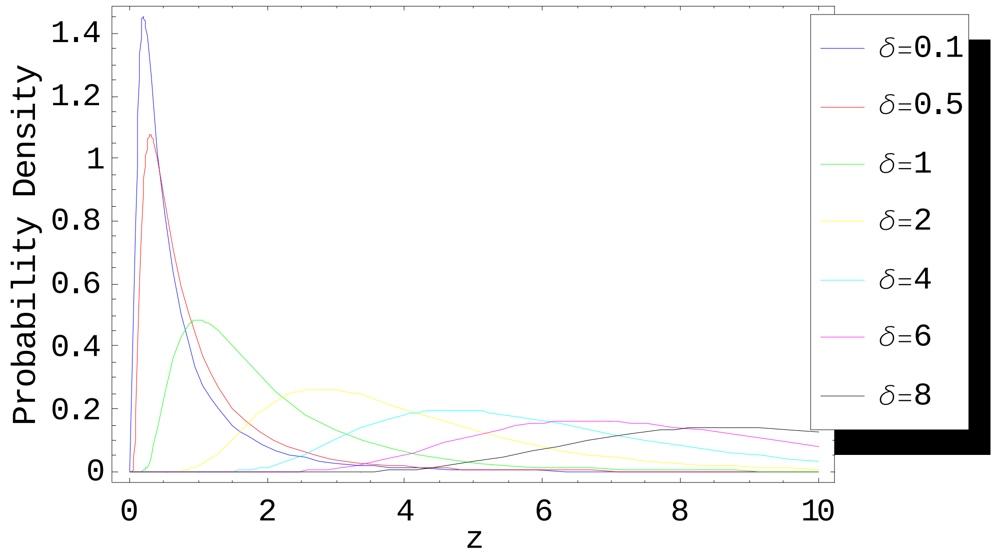
$$r_3 \equiv E[Z^3] = \frac{\delta(3+3\delta\gamma+\delta^2\gamma^2)}{\gamma^5},$$

$$r_4 \equiv E[Z^4] = \frac{\delta(15+15\delta\gamma+6\delta^2\gamma^2+\delta^3\gamma^3)}{\gamma^7}.$$

Note that the form of centered moments of (10) tells us that the IG probability density is always positively skewed and the excess kurtosis is always positive. Figure 2 illustrates the shape of the IG distribution with varying parameters. In Panel A, as γ increases holding δ constant, all the standardized moments decrease. In Panel B, as δ rises holding γ constant, the mean and variance rise while the skewness and excess kurtosis fall.



A) δ = 1 and varying γ



B) Varying δ and γ = 1

Figure 2 Plot of IG probability density

Probably, the most important property of the IG distribution is its infinite divisibility. Let Z be an IG random variable with $\mu \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$. Then, there exist n pieces of $i.i.d.$ random variable $Z_1, Z_2, ..., Z_n$ each from the IG distribution with $\mu / n \in \mathbb{R}^+$ and $\lambda / n \in \mathbb{R}^+$ such that:

$$Z \underline{\underline{d}} Z_1 + Z_2 + ... + Z_n ,$$

which is the definition 3.3 of Matsuda (2005). This indicates that the IG distribution generates a class of increasing Lévy processes (subordinators).

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